Given $\begin{pmatrix} f_0 \\ \vdots \\ f_{n-1} \end{pmatrix} \in \mathbb{C}^N$, then the DFT is defined as:

$$\vec{c} = \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix} \in \mathbb{C}^N, \text{ where } c_k = \frac{1}{n}\sum_{j=0}^{n-1} f_j e^{-i\frac{2jk\pi}{n}}, \text{ for } k = 0, \cdots, n-1.$$

The inverse DFT is defined as:

$$f_j = \sum_{k=0}^{n-1} c_k e^{i\frac{2jk\pi}{n}}$$

Properties of DFT:

- Given $\{f_i\}_{i=0}^{n-1}$ and $\{g_i\}_{i=0}^{n-1}$,

$$\widehat{c_1 f + c_2 g} = c_1 \hat{f} + c_2 \hat{g}$$

- Plancherel theorem:

$$\sum_{k=0}^{n-1} |f_j|^2 = n \sum_{k=0}^{n-1} |c_k|^2$$

Proof: $n \sum_{k=0}^{n-1} |c_k|^2 = n \sum_{k=0}^{n-1} \left| \frac{1}{n} \sum_{j=0}^{n-1} f_j e^{-i\frac{2jk\pi}{n}} \right|^2$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j,l=0}^{n-1} f_j \bar{f_l} e^{-i\frac{2(j-l)k\pi}{n}}$$

$$= \underbrace{\frac{1}{n} \cdot n \sum_{j=0}^{n-1} f_j \bar{f_j}}_{\sum_{j=0}^{n-1} |f_j|^2} + \underbrace{\frac{1}{n} \sum_{j,l=0}^{n-1} \sum_{k=0}^{n-1} f_j \bar{f_l} e^{-i\frac{2(j-l)k\pi}{n}}}_{\text{Note that } \sum_{k=0}^{n-1} e^{-i\frac{2(j-l)k\pi}{n}} = \frac{1-\left(e^{-i\frac{2k\pi}{n}}\right)^n}{1-e^{-i\frac{2k\pi}{n}}} = 0}$$

$$= \sum_{k=0}^{n-1} |f_j|^2$$

- Periodicity of DFT.

Define $f_t = f_{t-kn}$, where $t-kn \in [\![0, n-1]\!]$, $\forall t \in \mathbb{Z}$.

Pick $m \in [\![0, n-1]\!]$, consider the vector $\begin{pmatrix} f_m \\ f_{m+1} \\ \vdots \\ f_{n+m-1} \end{pmatrix}$ $\begin{matrix} g_0 \\ g_1 \\ \vdots \\ g_{n-1} \end{matrix}$

Let us calculate the DFT $\begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix}$ of this vector.

$$c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_{m+j} e^{-i\frac{2jk\pi}{n}} = \frac{1}{n} \sum_{j=0}^{n-1} f_{m+j} e^{-i\frac{2(m+j)k\pi}{n}} \cdot e^{i\frac{2mk\pi}{n}}$$

$= e^{i\frac{2mk\pi}{n}} c_k'$, where c_k' is the DFT of $\begin{pmatrix} f_0 \\ \vdots \\ f_{n-1} \end{pmatrix}$.

Def: Given $\{f_i\}_{i=0}^{n-1}$, $\{g_i\}_{i=0}^{n-1}$,

define $(f * g)_i := \sum_{k=0}^{n-1} f_k g_{i-k}$

Prop: $(\widehat{f*g})_k = n \hat{f}_k \hat{g}_k$.

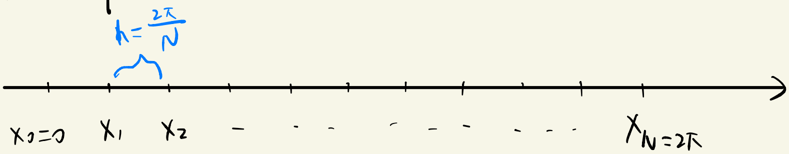
Proof: HW2.

- Average value of a vector.

$$\bar{f} = \frac{1}{n} \sum_{j=0}^{n-1} f_j = \frac{1}{n} \sum_{j=0}^{n-1} f_j e^{-i\frac{2j\cdot 0\cdot\pi}{n}} = \hat{f}_0.$$

Example: Solve the following ODE: $\frac{d^2u}{dx^2} - 2\frac{du}{dx} = f(x)$, $x \in [0, 2\pi]$.

First, partion the interval $[0, 2\pi]$ into N subintervals:



Assume $\vec{u} = \sum_{k=0}^{N-1} \lambda_k \overrightarrow{e^{ikx}}$.

Use central limit scheme,

$$\left(\frac{d^2\vec{u}}{dx^2}\right)_k = \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2}, \quad \left(\frac{d\vec{u}}{dx}\right)_k = \frac{u_{k+1} - u_{k-1}}{2h}$$

Define
$$D_1 = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & 0 \cdots & 0 \\ & & & & \\ 1 & \cdots & & \cdots & -2 \end{pmatrix}, \quad D_2 = \frac{1}{2h}\begin{pmatrix} 0 & 1 & \cdots & \cdots & -1 \\ -1 & 0 & 1 & \cdots & \cdots \\ \vdots & & & & \\ 1 & \cdots & \cdots & -1 & 0 \end{pmatrix}.$$

$$D_1 \overrightarrow{e^{ikx}} = \frac{-4\sin^2\frac{kh}{2}}{h^2} \overrightarrow{e^{ikx}}$$

$$D_2 \overrightarrow{e^{ikx}} = \frac{i\sin(hk)}{h} \overrightarrow{e^{ikx}}.$$

$$\frac{-4\sin^2\frac{kh}{2}}{h^2} \cdot \lambda_k - 2\frac{i\sin(hk)}{h}\lambda_k = \hat{f}_k$$

$$\Rightarrow \lambda_k = \cdots\cdots.$$